

IMPLEMENTING AN OPTION-THEORETIC CMO VALUATION MODEL WITH RECENT PREPAYMENT DATA

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In another article we present a model for analyzing collateralized mortgage obligations (CMOs) that is embedded in an option-theoretic framework with “optimal prepayments” and “transaction costs of refinancing” (McConnell and Singh [1994]). As with other types of models for analyzing CMOs, ours requires empirical estimation of certain parameters of a prepayment process.

In this article, we describe a procedure for estimating the required parameters using only recent prepayment data. We focus on the use of only recent prepayment data because experience during 1992-1995 suggests that prepayment behavior can be subject to change after a pool has been outstanding for as little as three or four years. The estimation procedure is designed to incorporate pool-specific data.

Once we describe our valuation model and the estimation procedure, we estimate the parameters of the prepayment process with collateral from eight different CMOs. We show that different sets of mortgage collateral that have similar observable characteristics — such as coupon rate and term to maturity — can give rise to different estimates of the pool-specific prepayment parameters. The different pool-specific parameters give rise to different tranche values for what are otherwise identical tranches.

I. AN OVERVIEW

The option-theoretic approach for analyzing generic mortgage-backed securities was pioneered by Dunn and McConnell [1981] and Dunn and Spatt [1986]. In general, this approach adopts the perspective that mortgagors “optimally” exercise their options to

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refinance whenever the benefits from doing so exceed the costs. The benefits of refinancing are the savings in interest achieved by rolling into a lower-rate loan. The costs include the direct and indirect costs of refinancing.

To capture the fact that not all mortgagors in a pool of otherwise homogeneous mortgages refinance at the same time, Johnston and Van Drunen [1988] extend the model to allow the costs of refinancing to vary across mortgagors. The result is that even though each mortgagor is exercising the option optimally, not all loans in a pool prepay at the same instant. The introduction of heterogeneous costs of refinancing that vary across mortgagors also captures the burnout effect in mortgage prepayments because those mortgagors confronted with the lowest costs of refinancing prepay earliest, with the result that prepayments in a pool of collateral that has undergone one interest rate cycle are less sensitive to a future rate decline.

Direct application of the option-theoretic approach for analyzing CMOs is problematical, however, because this approach involves the use of a backward finite difference solution technique for determining the value of the mortgage securities. Of course, the cash flows of a CMO tranche depend not only on the exercise of the prepayment options at each instant in time, but also upon the history of all prior prepayments. Unfortunately, the backward finite difference solution technique does not allow for a memory of prior prepayments.

Thus, CMO analysis typically relies upon forward-looking Monte Carlo simulation analysis together with an empirically estimated prepayment function. Estimation of the prepayment function is sometimes described as curve-fitting, and, although it is often excellent and very useful curve-fitting, the empirically estimated prepayment functions do not directly model optimal exercise of the mortgagor's refinancing option.

In the spirit of Dunn and McConnell [1981], Dunn and Spatt [1986], and Johnston and Van Drunen [1988], our CMO model does allow for the optimal exercise of the prepayment option. The trick is that our approach involves a two-step solution procedure in which the first step uses a backward finite difference procedure to determine a level of the interest rate at which it is optimal for a given "category" of mortgagors to exercise its refinancing option at each time. A category of mortgagors is identified by the magnitude of the refinancing costs that it confronts. The model allows for as many refinancing cost categories as

desired, and a critical interest rate at which it is optimal for that cost category to refinance at each point in time — called the refinancing boundary — is determined for each refinancing cost category.

In the second step, a forward-looking Monte Carlo simulation is used to generate paths of interest rates. If the interest rate from the Monte Carlo procedure hits the refinancing boundary for a particular category of mortgagors, that category is eligible to prepay. Even within a given cost category, however, some mortgagors respond with a delay when the interest rate falls below the threshold level. To capture this delayed response, the model allows for a "lag" in prepayments across mortgagors within a given cost category. Finally, the model allows for a "background" level of prepayments that occurs even when it is not optimal for any class of mortgagors to refinance. The background prepayments reflect prepayments due to relocations and mortgage defaults that occur independent of the level of interest rates.

Implementation of our CMO model requires specification of a model of the term structure of interest rates and estimation of the fraction of the mortgagors in a pool that comes from each category of refinancing costs, estimation of the lag in the prepayment response function, and estimation of the parameters that determine the background level of non-interest rate-related prepayments.

In our original presentation of the model, we arbitrarily specified what we thought to be reasonable parameters of the prepayment process to illustrate the valuation model. In this article, we present a procedure for estimating the fraction of mortgagors from each cost category and for estimating the lag in the prepayment response function. The procedure that we present can be implemented using only recent prepayment data.

Our motivation for focusing on recent prepayment data is that recent prepayment experience suggests that, in the aggregate, the mortgage market has undergone a structural shift in mortgagors' responsiveness to refinancing opportunities. For example, both the 1992-1993 and the 1986-1987 time periods witnessed a drop in the mortgage rate of roughly 400 basis points in a relatively short period of time, but prepayment speeds were much higher in the later period than in the earlier one (Breedon [1994]).

Structural shifts that may have led to a change in mortgagors' responsiveness to refinancing opportunities include declining out-of-pocket costs of refinancing,

increased competition among mortgage originators, and greater sophistication on the part of borrowers in exercising their prepayment options. Such structural shifts could easily mean that mortgagors within a pool shift from one cost category to another during the life of the pool. By using only recent prepayment data, the procedure that we propose and illustrate here accommodates shifts in mortgagors' propensity to exercise their prepayment options.

Suppose, for example, that a pool has been outstanding for four years, but the analyst has reason to believe that a shift in mortgagors' sensitivity to interest rates has occurred over the last twelve months. In that case, the parameters that determine the fraction of the collateral from each refinancing cost category, along with the lag parameter, can be estimated with only the most recent twelve months of data. Because the rate of the background level of prepayments is assumed to be largely independent of the level of interest rates, this parameter can be estimated with a longer time series, although updating may be appropriate if the analyst has reason to believe that a shift has occurred in this rate as well.

Our estimation procedure can be thought of as having three steps. In the first step, the backward finite difference procedure is used to determine the critical boundary of interest rates for each cost category of mortgagors. To implement this step, mortgagors are assumed to come from a discrete number of refinancing cost categories; and for each category, the cost of refinancing is specified as a fixed proportion of the remaining principal balance of the loan.

In the second step, a quadratic optimization procedure is used to estimate the fraction of the collateral in the pool that comes from each refinancing cost category. To implement the second step, an arbitrary value is assigned to the lag response parameter.

The third step involves an iteration in which a different value is assigned to the lag parameter, and the optimization procedure in step two is repeated. At each iteration, the sum of squared errors is computed. The iteration procedure is repeated until a minimum sum of squared errors is determined. At that point, the model is fully estimated.

The output of the estimation procedure is an estimate of the fraction of the remaining collateral from each refinancing cost category for each pool and an estimate of the parameter that determines the lag in the responsiveness of mortgagors to refinancing opportuni-

ties. A single lag parameter is estimated for all the pools used in the estimation procedure.

II. THE VALUATION MODEL

Our valuation model assumes that each refinancing cost category i is characterized by a refinancing cost RF_i that is proportional to the remaining principal balance of the loans in that cost category. To illustrate the model, assume that the risk-adjusted short-term interest rate r follows the stochastic process

$$dr = [\phi(t) - \alpha(t)r]dt + \sigma(r, t)dz \quad (1)$$

where dr represents the change in r over the time interval dt ; $\phi(t) - \alpha(t)r$ is the time-dependent mean-reverting drift of r ; $\sigma(r, t)$ is the instantaneous volatility of r ; and dz is an increment of a standard Weiner process.

A mortgage-backed security has two values: the value of the cash flows to the holder of the security, and the value of the cash flows from the perspective of the mortgagors. Because of the cost of refinancing, these two cash flow streams have different values.

Let M_i be the value of the cash flows to the investor from cost category i (i.e., the value of the MBS collateralized by mortgages from category i); V_i be the value of the cash flows from the perspective of the mortgagor (i.e., the value of the mortgagors' liabilities); A_i be the continuous principal and interest cash flow from the mortgages in cost category i ; F^i be the remaining principal balance of the mortgages in cost category i ; and c_s be the proportional servicing and guarantee fee of the MBS. Given (1), and invoking the fundamental valuation equation, the value of an MBS collateralized by mortgages from cost category i is governed by a system of equations

$$\begin{aligned} \frac{1}{2} \sigma(r, t)^2 V_{rr}^i + (\phi(t) - \alpha(t)r) V_r^i \\ + V_t^i + A^i - r V^i = 0 \end{aligned} \quad (2A)$$

$$\begin{aligned} \frac{1}{2} \sigma(r, t)^2 M_{rr}^i + (\phi(t) - \alpha(t)r) M_r^i \\ + M_t^i + (A^i - c_s F^i(t)) - r M^i = 0 \end{aligned} \quad (2B)$$

Assuming that mortgagors evaluate their prepay-

ment options with no delay, the prepayment boundary condition for mortgagors in cost category i is $V^i(r, t) \leq (1 + RF^i)F^i(t)$, and the boundary condition for the value of the MBS supported by mortgages from cost category i is $M^i(r, t) = F^i(t)$ whenever $V^i(r, t) = (1 + RF^i)F^i(t)$.

The first step in our CMO valuation model uses a finite difference solution procedure to determine the refinancing boundary of the risk-adjusted short-term rate, $r_c^i(t)$, for which it is optimal for mortgagors in cost category i to refinance their loans at time t . Whenever the risk-adjusted short-term rate falls below this level, all the mortgages in cost category i are assumed to prepay. A critical refinancing boundary is determined for each cost category.

The critical refinancing boundaries of interest rates are then employed in the second step of the valuation procedure along with a Monte Carlo simulation to value the tranches of the CMO. Additionally, however, the second step incorporates the idea that mortgagors respond to the decline in rates with a delay. That is, even though it is "optimal" to prepay, some mortgagors delay their decision to do so — perhaps because mortgagors evaluate their options only at discrete intervals. The second step also incorporates the empirical observation that some mortgagors prepay even when it is not optimal to refinance. These prepayments are captured by the background level of prepayments that occurs independent of the level of interest rates.

Let the continuous rate $\mu(t)$ represent the background level of prepayments so that at any given time, the rate of non-interest rate-related prepayments is $\pi_b(t)$ such that $\pi_b(t)dt = 1 - e^{-\mu(t)dt}$. Let ρ represent the parameter associated with the delay in the refinancing decision that occurs when the risk-adjusted short-term rate hits a refinancing boundary.¹

Whenever it is optimal to refinance, the prepayment probability for cost category i is $\pi_r^i(t) dt = 1 - e^{-(\mu(t)+\rho)dt}$. $1/\rho$ gives the half-life of mortgages whenever it is optimal to refinance. Note that $\mu(t)$ and ρ are assumed to be the same for each refinancing cost category.

Then, for refinancing cost category i , the rate of prepayments is $\pi^i(t) = \pi_b(t)$ whenever the short-term rate is above the refinancing boundary $r_c^i(t)$, and it is $\pi^i(t) = \pi_r^i(t)$ whenever the short-term rate falls below the refinancing boundary for that cost category.

CMO valuation requires allocation of the cash flows from the underlying collateral to the various tranches according to the structure of the deal. Because

these cash flows are path-dependent, CMO valuation requires that the cash flows from the various cost categories be tracked through time. The cash flows from these various cost categories are aggregated, and the aggregate cash flows are allocated to the appropriate tranches at each time.

To do this, let w^i denote the initial fraction of the pool with refinancing cost RF^i ; I be the number of refinancing cost categories; and G denote the underlying generic collateral so that $A^i = w^iA$, and $F^i(t) = w^iF_G(t)$. Let the fraction of the collateral from cost category i surviving until time t be $S^i(t)$, and let changes in this fraction be

$$dS^i(t) = -\pi^i(t)S^i(t)dt \quad (3)$$

where the dependence of $S^i(t)$ and $\pi^i(t)$ on r has been suppressed for notational convenience. The fraction of the aggregate collateral that survives until time t is then

$$S(t) = \sum_{i=1}^I w^i S^i(t) \quad (4)$$

and the prepayment rate on the declining balance of the aggregate collateral is

$$PR(t) = \sum_{i=1}^I w^i \pi^i(t) S^i(t) \quad (5)$$

The aggregate cash flow from the collateral at time t is

$$CF_G(t) = S(t)A + PR(t)F_G(t) - S(t)c_s F_G(t) \quad (6)$$

It is this aggregate cash flow that is allocated to the various tranches. Tranche valuation is performed by taking the average of the discounted value of the cash flows for that tranche across all interest rate paths. The valuation procedure is based on the risk-neutral valuation equation

$$M(r, t) = E_t \left[\int_t^T \left\{ CF(s) e^{-\int_t^s r(v)dv} \right\} ds \right] \quad (7)$$

where E_t denotes expectations taken with respect to the equivalent martingale measure or risk-adjusted process,

T is the maturity of the security, and CF(t) is the cash flow to the tranche.

To summarize, in the first step of our valuation model, a backward finite difference procedure is used to determine the critical boundary of interest rates for each refinancing cost category. In the second step, the Monte Carlo procedure projects risk-adjusted interest rate paths, which, in turn, are compared with the critical interest rates for each refinancing cost category of mortgagors for each remaining term to maturity of the collateral.

This comparison in conjunction with the lag in the refinancing response rate and the background prepayment rate determines prepayments along each path for each cost category of mortgagors. These prepayments are then aggregated by using Equation (5), and the cash flows as determined by Equation (6) are allocated to the tranches. Tranche valuation is then performed according to Equation (7).

III. ESTIMATING THE PREPAYMENT PARAMETERS WITH A QUADRATIC OPTIMIZATION PROCEDURE

Implementation of our valuation model requires specification of a model of the term structure of interest rates and estimation of the background level of prepayments $\mu(t)$, the lag response parameter ρ , and the fraction of the collateral (as of today) from each refinancing cost category. To estimate the fraction of the collateral from each refinancing cost category and the lagged response parameter ρ , we take as given that the parameters of the functions that determine the background level of prepayments have been estimated. For example, the background level of prepayments may be estimated as a function of the season of the year, the age of the pool, and other non-interest rate data.

Monthly prepayment data are used in the estimation. The monthly prepayment factor for each pool is the single month mortality (SMM), which is calculated as the dollar amount of prepayments in the month divided by the dollar amount of principal outstanding at the beginning of the month.

The number of cost categories can be any arbitrarily large number. The categories are identified by their prespecified proportional costs of refinancing. Let I be the number of cost categories. The Ith category is assumed to confront an infinite cost of refinancing, so

that there is no interest rate for which this category of mortgagors will refinance.

For implementation of the estimation procedure, we define the fraction of the collateral from each cost category at the *beginning* of the time series of data as $X_1, X_2, X_3, \dots, X_I$. For example, if we use the most recent twelve months of prepayment data to estimate the model, $X_1, X_2, X_3, \dots, X_I$ are the fractions of the collateral from each of the cost categories at the beginning of this twelve-month period, and $X_1 + X_2 + X_3 + \dots + X_I = 1$.

We define the fraction of the collateral from each of the I cost categories at the *current* time to be $Y_1, Y_2, Y_3, \dots, Y_I$ and $Y_1 + Y_2 + Y_3 + \dots + Y_I = 1$. Recall that we define the fraction of the collateral from each cost category that remains outstanding at any month, t, as $S^1(t), S^2(t), S^3(t), \dots, S^I(t)$. Let the first month in the time series be demarcated as $t = 0$ so that $S^1(0) = S^2(0) = S^3(0) = \dots = S^I(0) = 1$, and the current month is denoted as K.

We define a dummy variable $D[i, t]$ for each refinancing cost category for each month in the time series of prepayment data. We determine the value of this dummy variable from the refinancing boundaries determined in the first step of the CMO valuation model. Specifically, the dummy variable $D[i, t]$ is set equal to 1 if the risk-adjusted short-term rate is less than or equal to the refinancing boundary for category i at time t as determined from the backward solution procedure; otherwise $D[i, t]$ is equal to 0.

Now, given the estimated function for $\mu(t)$ and noting that $S^i(0) = 1$, we use an arbitrarily assumed (but reasonable) initial value for ρ , and, for each month t, for each cost category i, we calculate the fraction of the collateral surviving as

$$S^i(t) = S^i(t-1)\exp(-\mu(t) - \rho D[i, t])\Delta t \quad (8)$$

where $\Delta t = 1/12$. The calculated values of $S^i(t)$ are, of course, contingent on the arbitrarily assumed initial value of ρ . They are therefore an intermediate step toward the calculation of the X_i s and, eventually, the Y_i s.

Let SMM(t) be the actual (observed) prepayment rate for the aggregate collateral underlying the CMO for any month t (i.e., the single month mortality rate). Let ESMM(t) be the expected prepayment rate for any month t for the aggregate collateral underlying the CMO. ESMM(t) is defined as

$$\begin{aligned}
 \text{ESMM}(t) = & 1 - \exp(-\mu(t) - \rho\{S^1(t-1)X_1D[1, t] + \\
 & S^2(t-1)X_2D[2, t] + S^3(t-1) \times \\
 & X_3D[3, t] + \dots + S^I(t-1)X_1D[I, t]\} / \\
 & S(t-1)) \Delta t
 \end{aligned} \tag{9}$$

where $S(t) = \prod_{s=1}^t [1 - \text{SMM}(t)]$. Note that because the cost of refinancing for category I is ∞ , $D[I, t] = 0$ for all t .

The values of X_i are estimated by means of an optimization program that minimizes the squared differences between the log of the actual proportion surviving each month, $\log[1 - \text{SMM}(t)]$, and the log of the expected proportion surviving in that month, $\log[1 - \text{ESMM}(t)]$. The optimization program is

$$\begin{aligned}
 \text{Min } f(X_1, X_2, X_3, \dots, X_I) = \\
 \sum_{t=1}^K [\log(1 - \text{SMM}[t]) - \log(1 - \text{ESMM}[t])]^2
 \end{aligned} \tag{10}$$

or, equivalently

$$\begin{aligned}
 \text{Min } \sum_{t=1}^K [\log(1 - \text{SMM}[t]) + (-\mu(t) - \\
 \rho\{S^1(t-1)X_1D[1, t] + S^2(t-1)X_2D[2, t] + \\
 S^3(t-1)X_3D[3, t] + \dots + \\
 S^I(t-1)X_1D[I, t]\} / S[t-1])\Delta t]^2
 \end{aligned}$$

subject to $X_1 + X_2 + X_3 + \dots + X_I = 1$ and $X_i \geq 0$ for $i = 1, 2, 3, \dots, I$, and where K is the number of months in the time series. The output of this optimization procedure is a set of $X_1, X_2, X_3, \dots, X_I$ for each pool of collateral. The optimization procedure iterates on ρ across as many different pools of collateral as desired until the sum of squared differences is minimized across all pools of collateral.

The final step in the estimation procedure is to calculate the fraction of the remaining collateral (as of the current date) from each refinancing cost category

for each pool of collateral as

$$Y_i = X_i S^i(K) / [X_1 S^1(K) + X_2 S^2(K) + X_3 S^3(K) + \dots + X_I S^I(K)] \tag{11}$$

where K corresponds to the most recent month of data.

IV. APPLICATION OF THE ESTIMATION PROCEDURE

Implementation of our valuation model and our estimation procedure requires specification of a model of the term structure of interest rates. Our model is general and can accommodate a wide variety of term structure models. For illustrative purposes, we use a version of the Heath, Jarrow, and Morton (HJM) [1992] model. The HJM approach starts with the dynamics of the short-term forward rate, $f(t, T)$, at time t for date T .

One example of a single-factor model with a specific volatility structure in the HJM approach models the forward rate dynamics as

$$df(t, T) = a(t, T) dt + \sigma e^{-(\lambda/2)(T-t)} dz \tag{12}$$

where z is the standard Weiner process. The instantaneous volatility of the current maturity forward rate, $f(t, t)$, is σ , which declines exponentially for forward rates of higher maturities. The parameter λ represents the rate at which the volatility declines with increasing maturity. Assuming the forward rate dynamics in (12), the short-term interest rate follows a mean-reverting process given as

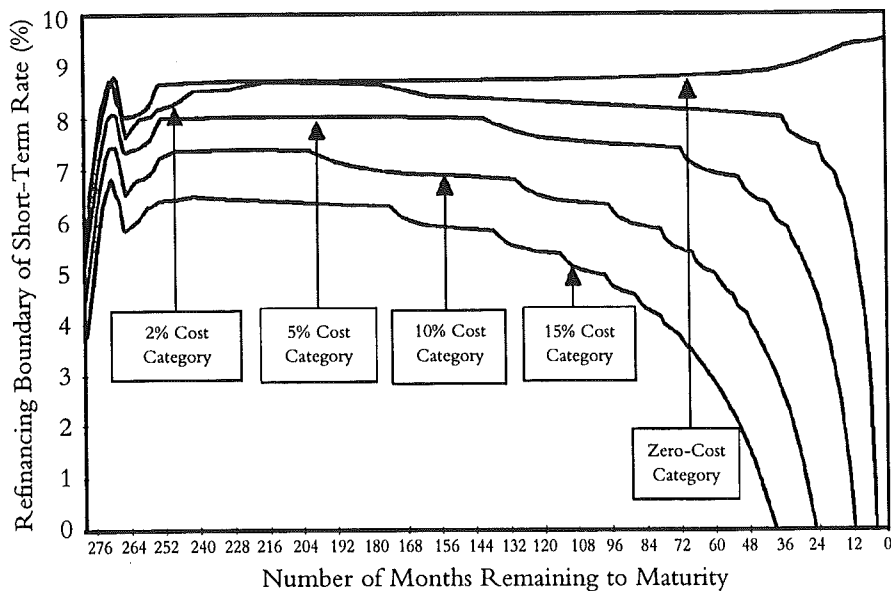
$$dr = \lambda/2[\eta(t) - r]dt + \sigma dz \tag{13}$$

where

$$\begin{aligned}
 \eta(t) = & (2/\lambda) \partial f(0, t) / \partial t + f(0, t) - \\
 & 2(\sigma/\lambda)^2 [1 - e^{-\lambda t}]
 \end{aligned}$$

As an example, if we choose $\lambda = 0.2$, the volatilities of the one-year maturity, the five-year maturity, the ten-year maturity, and the twenty-year maturity zero-coupon bonds are, respectively, 0.952σ , 0.787σ , 0.632σ , and 0.432σ . In the fundamental valuation equations (2A) and (2B), the HJM model implies $\phi(t) =$

EXHIBIT 1 ■ Refinancing Boundaries for 9.5% WAC, 280 Months WAM Mortgages on December 31, 1994



sonable representation of the volatility structure that was experienced during 1994. Five refinancing cost categories are specified for each pool with proportional refinancing costs of 2%, 5%, 10%, 15%, and ∞ . The last category, of course, experiences no refinancing.

The backward finite difference procedure is used to determine the critical boundary of interest rates for each cost category for each pool of collateral. Because the terms to maturity of the different pools of collateral are very similar, and because they have long remaining terms to maturity, the refinancing boundaries are very similar across the pools.

Exhibits 1 and 2 give the

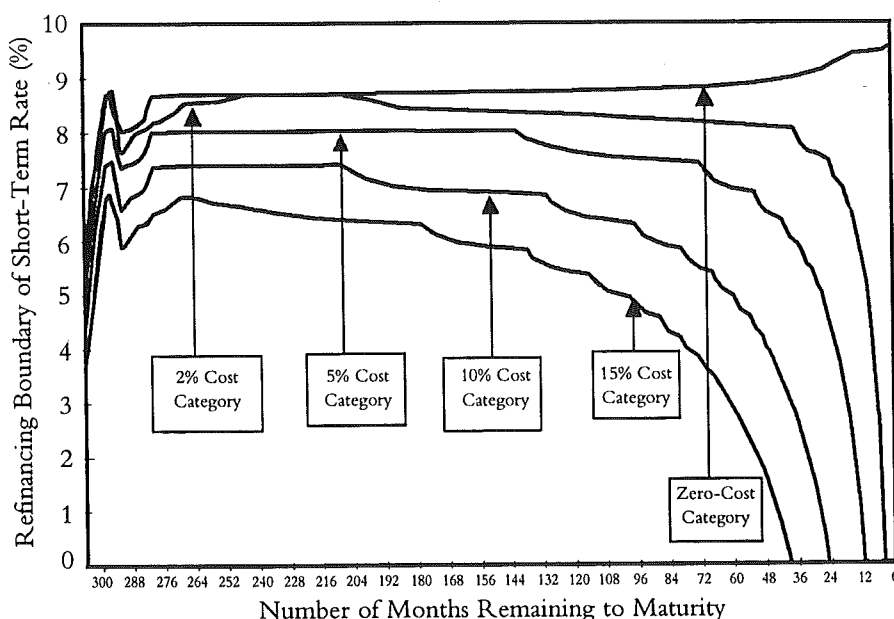
$$(\lambda/2)\eta(t) \text{ and } \alpha(t) = \lambda/2.^2$$

To estimate ρ and the $Y_{i,t}$ s, we use twelve months of prepayment data for eight different CMO deals, each of which is supported by Fannie Mae mortgages with a gross weighted-average coupon (WAC) of 9.5%. The twelve months of data include all of 1994. As of the end of 1994, the pools had been outstanding for fifty-four to eighty months so that the weighted-average maturities (WAMs) ranged from 306 to 280 months.

For each month of prepayment data, the HJM model is used to determine the term structure dynamics with the yields of fourteen different zero-coupon securities taken from the end of each month. These securities range from the one-month T-bill through the thirty-year T-bond. A value of $0.15r$ (15% of short-term rate) is used for σ , and a value of 0.2 is used for λ . These choices provide a rea-

sonable representation of the volatility structure that was experienced during 1994. Exhibit 1 is for the pool of collateral with the shortest WAM (280 months), and Exhibit 2 is for the pool of collateral with the longest WAM

EXHIBIT 2 ■ Refinancing Boundaries for 9.5% WAC, 306 Months WAM Mortgages on December 31, 1994



(306 months). The graphs also display the refinancing boundary for mortgagors who can refinance costlessly.

A few points should be mentioned. First, there is no refinancing boundary for the infinite-cost category, as there is no positive rate at which mortgagors in this category will refinance. Second, for the zero-cost category, the refinancing boundary (i.e., the short-term risk-adjusted rate) gradually moves closer to the coupon rate of the collateral as the remaining term to maturity becomes shorter.

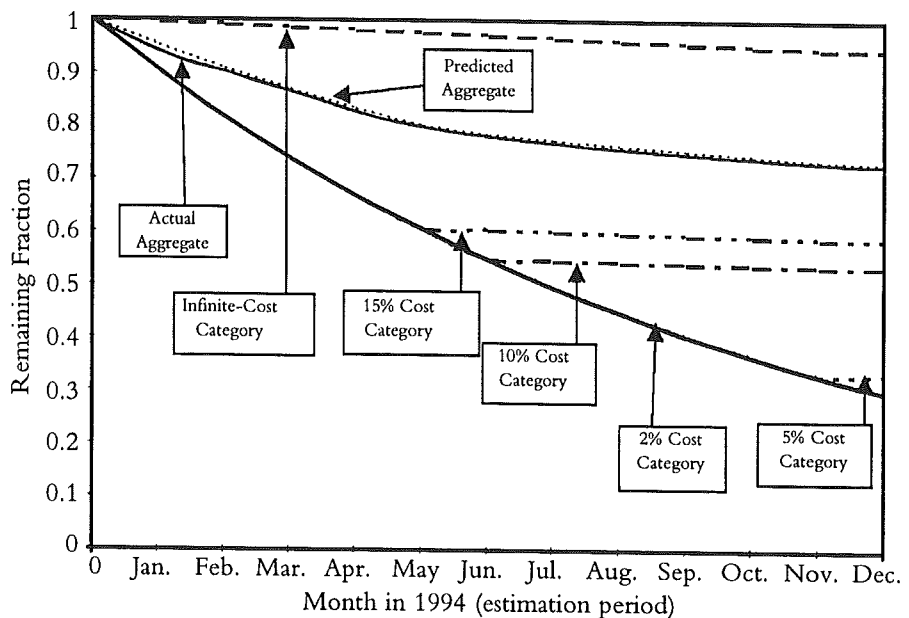
Third, for mortgagors who confront a positive, but finite, cost of refinancing, the refinancing boundary drops sharply as the remaining term to maturity of the collateral becomes shorter. This phenomenon occurs because the time period over which interest cost savings are generated by refinancing becomes shorter; ergo, the rate must fall farther to justify refinancing.

To implement the optimization/estimation procedure, a set of refinancing boundaries are generated for each month of prepayment data beginning with January 1994. These boundaries are used as inputs to estimate the dummy variables $D[i, t]$ that determine whether a particular cost category i is eligible to refinance at each month t . The dummy variables are then used in the optimization procedure to estimate the X_i 's and Y_i 's for each pool of collateral.

To determine total prepayments, we also require a background level of prepayments $\mu(t)$ and an estimate of the prepayment delay parameter ρ . Although the choice of an initial value for ρ makes no difference, we use $\rho = 1$ to begin. For simplicity, we assume a constant level for $\mu(t)$ of 6% per year.

The optimization procedure yields an estimated ρ of 1.15, which implies that, on average, once the interest rate hits the refinancing boundary for a particular cost category, roughly one-half the mortgages in that category will refinance within the next 1/1.15

EXHIBIT 3 ■ Predicted End-of-Month Remaining Fractions of Mortgages for Five Different Refinancing Cost Categories and Corresponding Actual and Predicted Values for Aggregate Collateral for 9.5% WAC and 280-Month WAM FNMA Collateral from January through December 1994



year. Exhibit 3 gives the actual fraction of the aggregate collateral remaining outstanding at the end of each month of 1994 for the pool with the shortest-maturity WAM.

The graph also gives the predicted fractions remaining for each refinancing cost category. Each category begins with a fraction of 1.0. Because of prepayments each month, the fraction of the collateral in that cost category that remains outstanding declines. Even the fraction of the collateral remaining in the ∞ cost category declines due to the background level of prepayments.

Notice that through Month 5, the four cost categories with finite refinancing costs decline at the same rate. This occurs because during early 1994, the refinancing rate was sufficiently low that all four finite cost categories were eligible to refinance. The divergence among the cost categories at Month 5 occurs because interest rates moved sharply upward during the second quarter of 1994. At that time, the higher cost categories became ineligible to refinance.

The fractions from Exhibit 3 are used to generate the X_i 's and Y_i 's for the collateral with the WAM of 292 months. Similar fractions are generated for each set

EXHIBIT 4 ■ Beginning of 1994 Proportions (X_i) and End of 1994 Proportions (Y_i) for Five Different Refinancing Cost Categories of Mortgages in Eight Different 9.5% WAC FNMA CMOs (estimated $\rho = 1.15$, assumed $\mu(t) = 6\%$ per year)

CMO Deal	Remaining WAM	Proportion of Aggregate Collateral	Cost Category				
			2%	5%	10%	15%	∞
1	280	Beginning Proportion	0.0888	0.0538	0.1109	0.2171	0.5295
		Ending Proportion	0.0364	0.0242	0.0807	0.1739	0.6848
2	280	Beginning Proportion	0.1180	0.0242	0.1154	0.2667	0.4727
		Ending Proportion	0.0498	0.0126	0.0866	0.2203	0.6306
3	284	Beginning Proportion	0.1055	0.0359	0.1042	0.3538	0.4005
		Ending Proportion	0.0461	0.0173	0.0810	0.3025	0.5531
4	286	Beginning Proportion	0.0859	0.0595	0.1226	0.3206	0.4113
		Ending Proportion	0.0374	0.0286	0.0949	0.2732	0.5659
5	286	Beginning Proportion	0.0754	0.1001	0.0815	0.3039	0.4391
		Ending Proportion	0.0326	0.0477	0.0627	0.2571	0.5999
6	292	Beginning Proportion	0.0829	0.1173	0.1131	0.3036	0.3831
		Ending Proportion	0.0374	0.0582	0.0907	0.2679	0.5458
7	295	Beginning Proportion	0.0966	0.0734	0.1093	0.3875	0.3332
		Ending Proportion	0.0443	0.0370	0.0890	0.3474	0.4823
8	306	Beginning Proportion	0.1066	0.0000	0.0568	0.5880	0.2486
		Ending Proportion	0.0498	0.0000	0.0471	0.5367	0.3664

of collateral and used to calculate the $X_{i,t}$ and $Y_{i,t}$ for them as well. These are given in Exhibit 4. Recall that $X_{i,t}$ is the proportion of the aggregate collateral from category i at the beginning of the time series (i.e., January 1, 1994), and $Y_{i,t}$ is the fraction of the aggregate collateral from category i at the end of the time series (i.e., December 31, 1994). The pools are ranked in Exhibit 4 from the CMO with the shortest WAM to the CMO with the longest WAM.

The table shows a significant degree of dispersion in the $X_{i,t}$ and $Y_{i,t}$ across the different CMOs. The dispersion in the $X_{i,t}$ means that the estimation procedure implies that the proportions of the collateral from the various cost categories differed as of the beginning of 1994. These differences are, of course, inferred from the differences in prepayment speeds during 1994.

Thus, even though each of the pools experienced the same refinancing cycle during 1994 and before, the estimation procedure implies different proportions for the cost categories at the end of 1994. These differences will, in turn, show up as differences in the tranche values across the CMO deals.

V. TRANCHE VALUATION

To illustrate the use of the estimation procedure

and our two-step CMO tranche valuation model, we use a simple five-tranche example with two different sets of collateral. This simple five-tranche example is hypothetical. It does not represent the structure of the actual deals supported by these sets of collateral.

The example CMO is composed of three sequential-pay tranches, a Z-bond, and an interest-only (IO) tranche. The A, B, and C sequential-pay tranches each constitute 30% of the principal, and the Z-bond constitutes the remaining 10%. The IO receives the differential interest between the interest paid on the A, B, C, and Z tranches and the interest payments received from the collateral. The interest rates on the tranches are 7.5%, 8.0%, 8.0%, and 8.0%, respectively. The generic collateral backing the deal has coupon rates of 9.5%.

The two sets of collateral used in the illustrations have WAMs of 292 months (i.e., Deal 6) and 295 months (i.e., Deal 7), respectively. These two sets of collateral are used because their observable characteristics are similar. Note, however, from Exhibit 4, that the two have different proportions in the different refinancing cost categories.

We value the tranches as of December 31, 1994, using the HJM term structure model, the observed

yield curve as of that date, an assumed short-term interest rate volatility of 15%, and $\lambda = 0.20$. We use $\rho = 1.15$ and assume $\mu(t) = 6\%$ per year. Finally, we assume that the generic MBS collateral supporting the deals and each of the tranches is priced so as to provide an option-adjusted spread (OAS) of 70 basis points.

The first row in Panel A of Exhibit 5 presents the value of the generic collateral supporting CMO Deal 6 and the values of the individual tranches for the hypothetical five-tranche CMO. The first row of Panel B gives the corresponding values for the collateral supporting CMO Deal 7.

The estimated values of the generic MBS in the two CMOs differ by \$0.08; i.e., \$105.14 versus \$105.06. The estimated values of tranches A, B, C, and Z have differences of -\$0.06, -\$0.19, \$0.00, and \$0.16 between the two CMOs. The difference in value of the IO is -\$0.14.

It is interesting to note that even though the value of the generic MBS in Deal 7 is lower than the corresponding value in Deal 6, tranches A and B have

a higher value in Deal 7 than in Deal 6. These observed differences are due to the differences in the starting proportions of the five refinancing cost categories between the two deals as of the valuation date, which, in turn, leads to different prepayment patterns along the same projected interest rate paths.

In the second and third rows of both Panel A and Panel B, the CMO tranche values and generic MBS values are presented for upward and downward shifts of 100 basis points in the yield curve. These values show significantly different sensitivities to changes in the yield curve between the two CMO deals. For example, for a 100-basis point upward shift in the yield curve, the value of the generic MBS decreases by \$4.35 (4.14%), and the value of the IO increases by \$0.59 (11.59%) in Deal 6. The corresponding values for Deal 7 are \$4.10 (3.90%) and \$0.75 (15.15%), respectively. The differences in estimated values and the yield curve sensitivities are meaningful, given that the pools of collateral underlying the two deals are quite similar.

EXHIBIT 5 ■ Tranche Valuation for CMOs Supported by the Collateral of CMO Deals 6 and 7 (valuation using HJM model of the term structure as of December 31, 1994)

Estimated $\rho = 1.15$

Assumed $\mu(t) = 6\%$ per year

Refinancing Cost Categories = 2%, 5%, 10%, 15%, and ∞

Face Value of Collateral = 100; Coupon Interest Rate = 9.5%; Service Fee = 0.5%

Face Value of Tranches: A = 30, B = 30, C = 30, Z-Bond = 10

Coupon Interest Rates of Tranches: A = 7.5%, B = 8.0%, C = 8.0%, Z-Bond = 8.0%

A. Valuation of Tranches with Collateral from CMO Deal 6 (maturity of collateral = 292 months)

(proportions of the five refinancing cost categories as of December 31, 1994, are 0.0374, 0.0582, 0.0907, 0.2679, and 0.5458)

Term Structure			Tranche Values					
Interest Rate Scenario	3-Month Rate	30-Year Rate	A	B	C	Z-Bond	Interest-Only	Generic MBS
Base Case	5.74	7.94	100.04	100.18	100.14	99.44	5.09	105.14
Up 100 bp	6.74	8.94	98.33	96.17	94.05	85.42	5.68	100.79
Down 100 bp	4.74	6.94	100.67	102.60	105.89	114.70	4.10	108.32

B. Valuation of Tranches with Collateral from CMO Deal 7 (maturity of collateral = 295 months)

(proportions of the five refinancing cost categories as of December 31, 1994, are 0.0443, 0.0370, 0.0890, 0.3474, and 0.4823)

Base Case	5.74	7.94	100.10	100.37	100.14	99.28	4.95	105.06
Up 100 bp	6.74	8.94	98.50	96.44	94.15	85.21	5.70	100.96
Down 100 bp	4.74	6.94	100.65	102.33	105.53	114.20	3.77	107.74

VI. SUMMARY AND CONCLUSION

In this article, we review our model for CMO valuation and present and implement a method for estimating the parameters of the prepayment process that underlies the model. Our implementation estimates the parameters with collateral from eight CMO deals. We show that the inclusion of pool-specific data in our estimation and valuation procedures can give rise to meaningful differences in tranche values even for very simple CMO structures.

ENDNOTES

¹Dunn and McConnell [1981] implicitly assume $\rho = \infty$. That is, there is no delay.

²The price $P(t, t + \tau)$ of a τ -maturity zero-coupon bond at time t is

$$y(\tau) = \frac{1}{\tau} \int_t^{t+\tau} f(t, s) ds$$

where $y(\tau)$ is the yield of the zero-coupon bond. Hence, the yield for a τ -maturity bond is given as

$$y(\tau) = \frac{1}{\tau} \int_t^{t+\tau} f(t, s) ds$$

With the process for the forward rate, the incremental change in the τ -maturity yield with an infinitesimal change in time is

$$dy(\tau) = \left[\frac{1}{\tau} \int_t^{t+\tau} a(t, s) ds - \frac{1}{\tau^2} \int_t^{t+\tau} f(t, s) ds \right] dt$$

$$+ \frac{2\sigma}{\lambda\tau} [1 - e^{-(\lambda/2)\tau}] dz$$

The volatility of longer-maturity bonds in relation to the instantaneous rate volatility of σ is given as

$$\frac{2\sigma}{\lambda\tau} [1 - e^{-(\lambda/2)\tau}]$$

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